**1.5. Newton’s method (method of tangents) for one nonlinear equation**

Let it be needed to solve the equation $f\left(x\right)=0$, if its separated root $c \in \left[a, b\right]$.

Newton’s iterations are $\left\{\begin{array}{c}x\_{0}\in \left[a,b\right] is arbitrary;\\x\_{k+1}=x\_{k}-\frac{f\left(x\_{k}\right)}{f\left(x\_{k}\right)},\\k=0,1,2,…\end{array}\right.$

**Theorem** (sufficient conditions for convergence). If

1. function $f\left(x\right)\in C^{2}\left[a,b\right]$ is continuous on the segment $\left[a,b\right]$ with its derivatives $f'(x)$ и $f''(x)$;
2. $∃M\_{1}>0 ∀x \in \left[a,b\right]$ $\frac{1}{\left|f'(x)\right|}<M\_{1}$;
3. $∃M\_{2}>0 ∀x \in \left[a,b\right] \frac{1}{2}∙\left|f^{''}\left(x\right)\right|< M\_{2}$;
4. $M= M\_{1}∙M\_{2}$;
5. $M∙\left|x\_{0}-c\right|<1$,

then Newton’s iterations converge to the exact solution *c* of the given equation, and besides the error of iteration *k* is estimated by the inequality:

$$\left|x\_{k}-c\right|<M^{-1}\left(M∙\left|x\_{0}-c\right|\right)^{2^{k}}$$

**Remark.** 1. The Newton’s method converges rather (fairly) quickly.

2. The Newton’s method converges at sufficiently close initial approximation to the root.

3. They execute sometimes the Newton’s method in the combination with some other method.

**Problem 3c.** Find the solution of the non-linear equation $e^{2\sin(x)}=x$ on the segment $\left[a,b\right]=[2,3]$ with the accuracy $ε= 10^{-4}$ by the Newton’s method.

*Solution.* $f\left(x\right)=e^{2\sin(x)}-x=0$;$f'\left(x\_{k}\right)=2\cos(x∙)e^{2\sin(x)}-1$. Newton’s iterations are $x\_{0}=\frac{a+b}{2}=2.5$; $x\_{k+1}= x\_{k}- \frac{f(x\_{k})}{f'(x\_{k})}$, *k*=0,1,… (see Table 1.5).

*Table 1.5*

|  |  |  |
| --- | --- | --- |
| *k* | $$x\_{k}$$ | $$f\left(x\_{k}\right)$$ |
| 0123 | 2.52.628502.635692.63571 | 0.809990.040640.000140.00000 |

Newton’s method Fortran program